Seismic coda due to non-linear elasticity

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Accepted 2007 September 29. Received 2007 September 25; in original form 2007 May 14

SUMMARY

Non-linear elastic response of rocks has been widely observed in laboratory, but very few seismic studies are reported in the literature, even though it is the most natural environment where this feature could be observed. Analytic solutions to the non-linear wave propagation phenomena are not readily available, and there is a need to use approximated techniques. It is clear that when a seismic wave propagates through a homogeneous non-linear elastic media, it will be perturbed by the non-linearity. This perturbation can be treated as a source of scattering, spreading the energy of the primary wave in space and time, contributing to the seismic coda. This is in some sense similar to the effect of heterogeneities. The properties of the coda due to the non-linearity depend on the amount of non-linearity and the seismic moment. Using a perturbation approach we calculate the amplitude of the scattered waves, and show that it can describe reasonably well the main features of real seismic codas.

Key words: Fault zone rheology; Theoretical Seismology; Wave Scattering and diffraction; Wave propagation.

1 INTRODUCTION

Seismology has shown that a linearly elastic response of rock provides a good description of a variety of phenomena related to earthquakes. Most observations related to large earthquakes can indeed be explained with the linear elastic theory, but it would be naive to believe that it provides a complete description, knowing that laboratory observations show that most rocks do behave non-linearly close to the rupture condition (Scholz 1990).

The classical approach to relate ground deformation to source parameters is through the representation theorem, and the principle of superposition. These are only valid for a linear constitutive law. Thus, if non-linear elasticity is important, one can expect that this classical approach is not appropriate.

Since non-linear elastic behaviour of most materials has been recognized as a significant factor for several centuries by physicists and material scientists (Bell 1984), it is important to investigate the effect of non-linear terms in the wave propagation phenomena, especially near fault zones. Among the studied features related to non-linear effects of rocks are the non-linear dependence of seismic velocities on stress (Birch 1961; Johnson & McCall 1994; McCall 1994), resonant peak shift (Winkler *et al.* 1979), effects of anisotropy and non-linearity (Zheng *et al.* 2006).

The acquisition of high precision displacements near sources has improved the understanding of sources of deformation in a wide range of scales of space and time, especially with respect to earthquake and volcanic processes. However, since the quantity and quality of observations are increasing rapidly, it provides the opportunity to increase the understanding of the source process, structure and rheology of the media, including second order effects, such as the non-linearity of the media.

In this paper, we consider a general non-linear constitutive relationship on the dynamic deformation near fault zones, showing that the effect of this non-linear term is to spread the seismic energy in space and time, even when the elastic media is homogeneous. We concentrate on the effect on secondary arrivals, calculating coda envelopes and show that it describes reasonable well the observations. Therefore, we suggest that a realistic analysis of coda waves requires the determination of non-linear elastic parameters as well as heterogeneities and intrinsic attenuation.

2 EQUATION OF MOTION

The constitutive relationship for a continuous media is known for the case of small deformations, when only linear terms are retained. Many possible generalizations to consider finite strain have been proposed (Truesdell 1966), but none completely satisfies all theoretical and observational considerations simultaneously. In the following we consider one possible generalization, and find solutions for it, with the aim of showing the range of implications the theory has, on aspects that can be observed seismically. The choice of the non-linear terms considered will change the form of the scattered waves and thus the coda, but would not change the general conclusion of this paper.

The constitutive equation for a continuous medium is given by $\tau = \frac{\partial W}{\partial \varepsilon}$, where W is the strain energy density. Considering only the dependence of the energy on the strain tensor, and assuming an isotropic media, the energy can be expressed in terms of the invariants of the strain tensor, $I_1 = \mathrm{Trace}(\varepsilon)$, $I_2 = \mathrm{Trace}(\varepsilon \cdot \varepsilon)$ and $I_3 = Det(\varepsilon)$, then, $W(\varepsilon) = W(I_1, I_2, I_3)$. The complete form of this function is not known, but certain restrictions apply. It is natural to expand W in series over the equilibrium state, which will be defined as $\varepsilon = 0$. The third power on ε, which gives rise to the Murnaghan coefficients (l, m, n) (Murnagham 1951), is not considered here because we require from physical arguments the energy W to be positive definite, and symmetric on strain. Third power coefficients do not satisfy these requirements. One can argue that the constitutive relationship should be established on the Cauchy stress tensor, which is referred to the instantaneous area of the surface element; or the Piola–Kirchhoff stress tensor, referred to the fixed undeformed media. In each case we will describe the same phenomena, but since the purpose of this paper is to show that the consideration of non-linear terms in the constitutive relationship implies a scattering phenomena even on a homogeneous medium, in the following we will consider a particular form of the strain energy, which is the most general up to fourth order given by

$$W(I_1, I_2, I_3) = \left(aI_1^2 + bI_2 + cI_1^4 + \mathrm{d}I_2^2 + eI_1^2 I_2\right)/2$$

from which the stress-strain relationship is given by:

$$\tau_{ij} = \left[aI_1 + 2cI_1^3 + eI_1 I_2\right]\delta_{ij} + \left[b + 2\mathrm{d}I_2 + eI_1^2\right]\varepsilon \tag{1}$$

and the elastodynamic equation $\rho\ddot{u} = \partial_j \tau_{ij} + f_i$ has the following expression:

$$\begin{aligned}
\rho\ddot{u}_i - a\varepsilon_{kk,i} - b\varepsilon_{ij,j} &= f_i + 6c\varepsilon_{pp}\varepsilon_{qq}\varepsilon_{kk,i} \\
&\quad + 2d(\varepsilon_{kl}\varepsilon_{lk}\varepsilon_{ij,j} + 2\varepsilon_{kl}\varepsilon_{lk,j}\varepsilon_{ij}) \\
&\quad + e(\varepsilon_{pq}\varepsilon_{qp}\varepsilon_{kk,i} + \varepsilon_{ll}\varepsilon_{kk}\varepsilon_{ij,j} + 2\varepsilon_{pp}\varepsilon_{lk}\varepsilon_{kl,i} \\
&\quad + 2\varepsilon_{pp}\varepsilon_{ij}\varepsilon_{kk,j}).
\end{aligned} \tag{2}$$

Hookes's law is obtained when $a = \lambda, b = 2\,\mu, c = d = e = 0$. From laboratory experiments (Scholz 1990), it is clear that $c, d, e < 0$. Exact analytic solutions to eq. (2) are not practical, and approximate solutions are required. One can assume that the constitutive relationship is approximately Hooke's law, where the slight departures from it, will perturb slightly the linear solutions, justifying a perturbation approach. In this approach, the non-linear terms are treated as a source of scattered waves.

2.1 Perturbation approach

The propagation of an incident P wave through a media where its constitutive relationship departs slightly from Hooke's law, is described by the elastodynamic equation (eq. 2). This equation does not admit wave solutions that can propagate without distortion (Bataille & Lund 1982). In general the incident wave is directly affected by the media. One way to compute this effect, is to assume that the solution is the superposition of the incident wave plus a small component due to the non-linearity, thus

$$\begin{aligned}
u &= u^{(l)} + u^{(nl)} \qquad u^{(nl)} \ll u^{(l)} \\
\varepsilon &= \varepsilon^{(l)} + \varepsilon^{(nl)} \qquad \varepsilon^{(nl)} \ll \varepsilon^{(l)}.
\end{aligned}$$

When strains are small, we can solve eq. (2) by iterations, where the first term is

$$\rho\ddot{u}_i^{(l)} - a\varepsilon_{kk,i}^{(l)} - b\varepsilon_{ij,j}^{(l)} = f_i$$

the well-known linear equation. The next term in the expansion is

$$\begin{aligned}
\rho\ddot{u}_i^{(nl)} - a\varepsilon_{kk,i}^{(nl)} - b\varepsilon_{ij,j}^{(nl)} &= 6c\varepsilon_{pp}^{(l)}\varepsilon_{qq}^{(l)}\varepsilon_{kk,i}^{(l)} + 2d\Big(\varepsilon_{kl}^{(l)}\varepsilon_{lk}^{(l)}\varepsilon_{ij,j}^{(l)} \\
&\quad + 2\varepsilon_{kl}^{(l)}\varepsilon_{lk,j}^{(l)}\varepsilon_{ij}^{(l)}\Big) + e\Big(\varepsilon_{pq}^{(l)}\varepsilon_{qp}^{(l)}\varepsilon_{kk,i}^{(l)} + \varepsilon_{pp}^{(l)}\varepsilon_{kk}^{(l)}\varepsilon_{ij,j}^{(l)} \\
&\quad + 2\varepsilon_{pp}^{(l)}\varepsilon_{lk}^{(l)}\varepsilon_{kl,i}^{(l)} + 2\varepsilon_{pp}^{(l)}\varepsilon_{ij}^{(l)}\varepsilon_{kk,j}^{(l)}\Big).
\end{aligned} \tag{3}$$

All the non-linear terms can be considered as sources of scattering. Calling these terms as $F^{(nl)}$ we have

$$\begin{aligned}
F_i^{(nl)} &= 6c\varepsilon_{pp}^{(l)}\varepsilon_{qq}^{(l)}\varepsilon_{kk,i}^{(l)} + 2d\Big(\varepsilon_{kl}^{(l)}\varepsilon_{lk}^{(l)}\varepsilon_{ij,j}^{(l)} + 2\varepsilon_{kl}^{(l)}\varepsilon_{lk,j}^{(l)}\varepsilon_{ij}^{(l)}\Big) \\
&\quad + e\Big(\varepsilon_{pq}^{(l)}\varepsilon_{qp}^{(l)}\varepsilon_{kk,i}^{(l)} + \varepsilon_{pp}^{(l)}\varepsilon_{kk}^{(l)}\varepsilon_{ij,j}^{(l)} + 2\varepsilon_{pp}^{(l)}\varepsilon_{lk}^{(l)}\varepsilon_{kl,i}^{(l)} \\
&\quad + 2\varepsilon_{pp}^{(l)}\varepsilon_{ij}^{(l)}\varepsilon_{kk,j}^{(l)}\Big).
\end{aligned} \tag{4}$$

2.2 Scattering solution

Once the force ($F^{(nl)}$) is determined, the solution for the displacement is given by the standard solution for a linearly elastic media (Aki & Richards 1980)

$$u_i(x,t) = \int\int G_{i,j}(x,x',t,t')F_j^{(nl)}(x',t')\,\mathrm{d}x'\,\mathrm{d}t', \tag{5}$$

where $G_{i,j}(x,x',t,t')$ is the Green's function.

To include the effect of absorption, or intrinsic attenuation, we add in eq. (5) the term ($\exp -\pi f t^*$) (Aki & Richards 1980), where f is the predominant frequency, $t^* = \int \frac{\mathrm{d}t}{Q}$ and Q is the quality factor.

To illustrate the effect of non-linearity we consider two simple cases: one, when the source is far from the region of non-linear behaviour, such that we can approximate it as a incident plane wave; and the other, when the source is embedded within the region of non-linear behaviour, such that we can consider it as a point source.

2.2.1 Incident plane wave

Assuming that a source is far from the region of interest, we can approximate the incident wave by a plane wave with a propagation direction defined by k_i, and displacement direction by a_i, where the displacement is given by

$$u_i^{(l)} = Aa_i \exp^{i(\omega t - k_i x_i)}$$

then

$$\begin{aligned}
u_{i,j}^{(l)} &= -iu_i k_j \\
\varepsilon_{ij}^{(l)} &= \frac{-i}{2}(u_i k_j + u_j k_i) \\
\varepsilon_{ll}^{(l)} &= -i(u \cdot k) \\
\varepsilon_{pq}^{(l)}\varepsilon_{qp}^{(l)} &= -\frac{1}{2}((u \cdot k)^2 + u^2k^2) \\
\varepsilon_{ij,j}^{(l)} &= -\frac{1}{2}\left(u_i k^2 + (u \cdot k)k_i\right) \\
\varepsilon_{jj}^{(l)}\varepsilon_{pp}^{(l)}\varepsilon_{qq,i}^{(l)} &= (u \cdot k)^3 k_i.
\end{aligned}$$

Thus the first non-linear term is

$$\begin{aligned}
F^{(nl)} &= 6c(u \cdot k)^3 k_i \\
&\quad + \frac{3d}{2}\left[(u \cdot k)^3 k_i + (u \cdot k)u^2k^2k_i + (u \cdot k)^2k^2u_i + u^2k^4u_i\right] \\
&\quad + \frac{3}{2}e\left[2(u \cdot k)^3 k_i + (u \cdot k)^2k^2u_i + (u \cdot k)u^2k^2k_i\right].
\end{aligned} \tag{6}$$

The main features are that the resulting force is proportional to (1) the amplitude of the incident wave up to the third power, (2) the wavenumber, or frequency, up to the fourth power. In comparison to scattering due to heterogeneities, which is linearly proportional to the amplitude of the incident wave, and proportional to the frequency up to third power.

These differences can be tested with seismological observations for special situations. For the particular case of a P wave, where $a_i = k_i$, eq. (7) becomes

$${}^P F^{(nl)} = 6(c + e + d)A^3k^4a_i \tag{7}$$

which is a vector in the direction of displacement of the P wave.

For the case of an S wave, where a_i is perpendicular to k_i, it becomes

$${}^S F^{(nl)} = \frac{3d}{2}A^3k^4a_i \tag{8}$$

which is a vector in the direction of displacement of the S wave.

At each point of space, the incident P or S waves, will produce a scattered P and S wave with a radiation pattern shown in Fig. 1, where the strength of the scattered waves are expressed in eqs (7) and (8).

2.2.2 Point source incident wave

When the source is embedded within the region of non-linear behaviour, it is important to solve first for the case of a point source. In this case the incident wave is expressed as

$$\begin{aligned}
u_i^{(l)}(x,t) &= \frac{1}{4\pi\rho\alpha^3 r}\gamma_i\gamma_j\dot{M}_{jk}\left(t - \frac{r}{\alpha}\right)\gamma_k \\
&\quad + \frac{1}{4\pi\rho\beta^3 r}(\delta_{ij} - \gamma_i\gamma_j)\dot{M}_{jk}\left(t - \frac{r}{\beta}\right)\gamma_k,
\end{aligned} \tag{9}$$

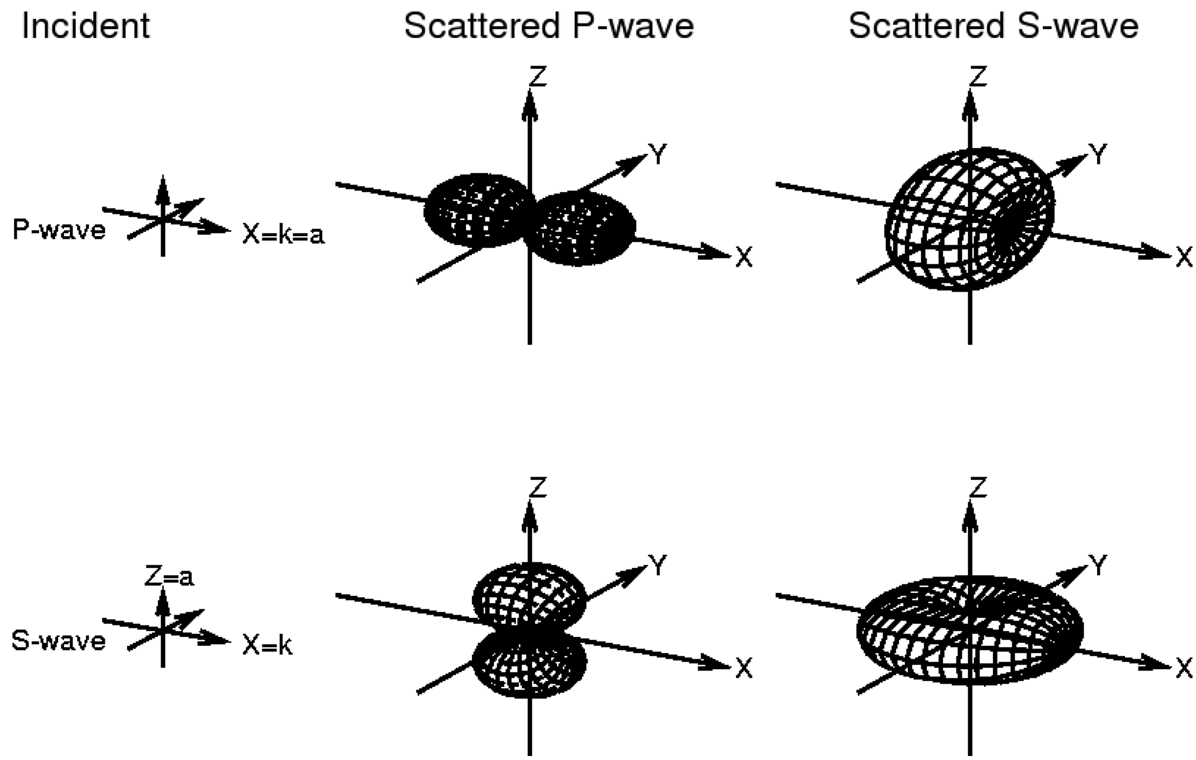

Figure 1. Radiation pattern of scattered waves due to incident P and S waves on a medium that is homogeneous but its constitutive relationship is non-linear given by eq. (1). For an incident P wave, the scattered P waves are mainly in the forward and backward directions. In the forward direction, its amplitude is negative, thus reducing the amplitude of the main wave. This would have a similar effect as absorption. The scattered S waves are mainly on the plane perpendicular to the incident P wave. For an incident S wave, the scattered P waves are in the direction of displacement of the incident wave, while the scattered S waves are mainly on the plane perpendicular to the displacement of the incident wave.

where $\gamma_i = r_i/r$, is the unit vector between the source and the observed point, while α and β, are the P and S waves, respectively, and M is the moment tensor of the source.

From eq. (9) we obtain

$$
\begin{aligned}
u^{(l)}_{i,j}(x,t) &= -\frac{1}{4\pi\rho\alpha^4 r}\gamma_i\gamma_j(\gamma\cdot\ddot{M}^{\alpha}\cdot\gamma)\\
&\quad -\frac{1}{4\pi\rho\beta^4 r}(\delta_{ik}-\gamma_i\gamma_k)(\gamma\cdot\ddot{M}^{\beta}\cdot\gamma)\\
\varepsilon^{(l)}_{ij}(x,t) &= -\frac{1}{4\pi\rho\alpha^4 r}\gamma_i\gamma_j(\gamma\cdot\ddot{M}^{\alpha}\cdot\gamma)\\
&\quad -\frac{1}{8\pi\rho\beta^4 r}\left[\ddot{M}^{\beta}_{il}\gamma_l\gamma_j+\ddot{M}^{\beta}_{jl}\gamma_i\gamma_l-2\gamma_i\gamma_j(\gamma\cdot\ddot{M}^{\beta}\cdot\gamma)\right]\\
\varepsilon^{(l)}_{ii}(x,t) &= -\frac{1}{4\pi\rho\alpha^4 r}(\gamma\cdot\ddot{M}^{\alpha}\cdot\gamma)\\
\varepsilon^{(l)}_{ij,j}(x,t) &= \frac{1}{4\pi\rho\alpha^5 r}\gamma_i(\gamma\cdot\dddot{M}^{\alpha}\cdot\gamma)\\
&\quad +\frac{1}{8\pi\rho\beta^5 r}\left[\dddot{M}^{\beta}_{il}\gamma_l-\left(\gamma\cdot\dddot{M}^{\beta}_{jl}\cdot\gamma\right)\gamma_i\right]\\
\varepsilon^{(l)}_{kk,i}(x,t) &= \frac{1}{4\pi\rho\alpha^5 r}\gamma_i(\gamma\cdot\dddot{M}^{\alpha}\cdot\gamma)\\
\varepsilon^{(l)}_{ij}\varepsilon^{(l)}_{ji}(x,t) &= \left(\frac{1}{4\pi\rho\alpha^4 r}\right)^2(\gamma\cdot\ddot{M}^{\alpha}\cdot\gamma)^2\\
&\quad +\frac{1}{2}\left(\frac{1}{4\pi\rho\beta^4 r}\right)^2\left[\gamma\cdot\ddot{M}^{\beta}\cdot\ddot{M}^{\beta}\cdot\gamma-(\gamma\cdot\ddot{M}^{\beta}\cdot\gamma)^2\right]\\
\varepsilon^{(l)}_{ik}\varepsilon^{(l)}_{ki,j}(x,t) &= \left(\frac{1}{4\pi\rho\alpha^4 r}\right)^2\left(\frac{-2}{\alpha}\right)(\gamma\cdot\dddot{M}^{\alpha}\cdot\gamma)(\gamma\cdot\ddot{M}^{\alpha}\cdot\gamma)\gamma_j\\
&\quad +\left(\frac{1}{4\pi\rho\beta^4 r}\right)^2\left(\frac{-1}{\beta}\right)\left[\gamma\cdot\dddot{M}^{\beta}\cdot\ddot{M}^{\beta}\cdot\gamma-(\gamma\cdot\ddot{M}^{\beta}\cdot\gamma)(\gamma\cdot\ddot{M}^{\beta}\cdot\gamma)\right]\gamma_j,
\end{aligned}
$$

where $M^\alpha = M(t - \frac{r}{\alpha})$ and $M^\beta = M(t - \frac{r}{\beta})$.

Inserting these terms into eq. (4), we obtain:

$$\begin{aligned}
F_i^{(nl)} = {} & \frac{6(c+d+e)}{\alpha}\left(\frac{1}{4\pi\rho\alpha^4 r}\right)^3 (\gamma\cdot\ddot{M}^\alpha\cdot\gamma)^2\gamma\cdot\dddot{M}^\alpha\cdot\gamma\gamma_i \\
& + \frac{2d+e}{2\beta}\left(\frac{1}{4\pi\rho\alpha^4 r}\right)^2 \frac{1}{4\pi\rho\beta^4 r}(\gamma\cdot\ddot{M}^\alpha\cdot\gamma)^2\left[\dddot{M}_i^\beta\cdot\gamma - (\gamma\cdot\dddot{M}^\beta\cdot\gamma)\gamma_i\right] \\
& + \frac{2d+e}{2\alpha}\frac{1}{4\pi\rho\alpha^4 r}\left(\frac{1}{4\pi\rho\beta^4 r}\right)^2 \gamma\cdot\dddot{M}^\alpha\cdot\gamma\left[\gamma\cdot\ddot{M}^\beta\cdot\ddot{M}^\beta\cdot\gamma - (\gamma\cdot\ddot{M}^\beta\cdot\gamma)^2\right]\gamma_i \\
& + \frac{d}{4\beta}\left(\frac{1}{4\pi\rho\beta^4 r}\right)^3 \left[(\gamma\cdot\ddot{M}^\beta\cdot\ddot{M}^\beta\cdot\gamma)(\dddot{M}_i^\beta\cdot\gamma - \gamma_i\gamma\cdot\dddot{M}^\beta\cdot\gamma)\right. \\
& + 2(\gamma\cdot\ddot{M}^\beta\cdot\dddot{M}^\beta\cdot\gamma)\left(\ddot{M}_i^\beta\cdot\gamma - \gamma_i\gamma\cdot\ddot{M}^\beta\cdot\gamma\right) - \dddot{M}_i^\beta\cdot\gamma(\gamma\cdot\ddot{M}^\beta\cdot\gamma)^2 \\
& \left. + (\gamma\cdot\dddot{M}^\beta\cdot\gamma)(\gamma\cdot\ddot{M}^\beta\cdot\gamma)\left[3\gamma_i(\gamma\cdot\ddot{M}^\beta\cdot\gamma) - 2\ddot{M}_i^\beta\cdot\gamma\right]\right] \\
& + \frac{e}{\beta}\left(\frac{1}{4\pi\rho\beta^4 r}\right)^2\left(\frac{1}{4\pi\rho\alpha^4 r}\right)\gamma_i(\gamma\cdot\ddot{M}^\alpha\cdot\gamma)[(\gamma\cdot\ddot{M}^\beta\cdot\dddot{M}^\beta\cdot\gamma) \\
& - (\gamma\cdot\dddot{M}^\beta\cdot\gamma)(\gamma\cdot\ddot{M}^\beta\cdot\gamma)] \\
& + \frac{e}{\alpha}\left(\frac{1}{4\pi\rho\alpha^4 r}\right)^2\left(\frac{1}{4\pi\rho\beta^4 r}\right)(\gamma\cdot\dddot{M}^\alpha\cdot\gamma)(\gamma\cdot\ddot{M}^\alpha\cdot\gamma)\left[\ddot{M}_i^\beta\cdot\gamma - \gamma_i\gamma\cdot\ddot{M}^\beta\cdot\gamma\right].
\end{aligned} \tag{10}$$

In the particular case of a source representing an explosion, $M_{ij} = M\delta_{ij}$, where δ_{ij} is the Kronecker symbol, we have

$$F_i^{(nl)} = \left(\frac{6(c+d+e)}{\alpha}\right)\left(\frac{1}{4\pi\rho\alpha^4 r}\right)^3 \ddot{M}^{\alpha^2}\dddot{M}^\alpha\gamma_i \tag{11}$$

which depends on r^{-3}, M^3, non-linear parameters, and points into the direction of γ_i. This means that: (1) from the point of view of the radiation pattern, the scattered *P* waves will be stronger in the forward and backward directions, (2) will produce scattered *S* waves, (3) since it decays fast (r^{-3}), the stations close to the source will detect this effect more effectively and (4) the amplitude depends on the third power of M.

2.2.3 *Comparison between non-linear and heterogeneous scattering*

For weak heterogeneities of the elastic parameters and density, the equivalent force representing the heterogeneities is a linear function of the moment and the variations of the elastic parameters and density, $F^{(he)} \sim \frac{\dddot{M}\cdot f(\delta\mu,\delta\lambda,\delta\rho)}{r}$, (Wu & Aki 1985). For density fluctuations alone, the radiation pattern of scattered *P* and *S* waves are the same as for non-linear elasticity (Fig. 1). The only difference between scattered waves produced by *density heterogeneities* and *non-linearities* relates to the strength of the scattered waves, which for the case of *non-linearities* depends on $\frac{\ddot{M}(t)^2\,\dddot{M}(t)}{r^3}$, while on *heterogeneities* on $\frac{\dddot{M}(t)}{r}$.

Thus the ratio of non-linear scattering over heterogeneous scattering is

$$\frac{u^{nl}}{u^{he}} \sim \left[\frac{\ddot{M}(t)}{r}\right]^2 \tag{12}$$

which refers to the time dependence, moment and distance. This can be used to differentiate both effects from real data sets.

To illustrate the difference between the effect of heterogeneity and non-linearity of the elastic media on the scattering properties, we compute the envelope of the scattered waves originated from an explosive point source, considering each case independently. For the distribution of heterogeneities we assume both a Gaussian and exponential autocorrelation function with different scale lengths, following (Wu & Aki 1985). For the non-linear case, the scattered waves are obtained using (11), for which the result is given by

$$\begin{aligned}
u_i^s(r,t) = {} & C_p\int\frac{\gamma_i\gamma_j\gamma_j'}{|r-r'||r'|^3}\ddot{M}^2\left(t-\frac{|r'|}{\alpha}-\frac{|r-r'|}{\alpha}\right)\dddot{M}\left(t-\frac{|r'|}{\alpha}-\frac{|r-r'|}{\alpha}\right)d^3r' \\
& + C_s\int\frac{(\gamma_i'-\gamma_i\gamma_j\gamma_j')}{|r-r'||r'|^3}\ddot{M}^2\left(t-\frac{|r'|}{\alpha}-\frac{|r-r'|}{\beta}\right)\dddot{M}\left(t-\frac{|r'|}{\alpha}-\frac{|r-r'|}{\beta}\right)d^3r'
\end{aligned} \tag{13}$$

which represent the propagation of a *P* wave from the source up to a point r' distributed through the volume, and then being scattered and propagating both as *P* and *S* waves, up to the point r, where

$$C_p = \frac{6(c+d+e)}{4\pi\rho\alpha^3}\left(\frac{1}{4\pi\rho\alpha^4}\right)^3$$

$$C_s = \frac{6(c+d+e)}{4\pi\rho\alpha\beta^2}\left(\frac{1}{4\pi\rho\alpha^4}\right)^3.$$

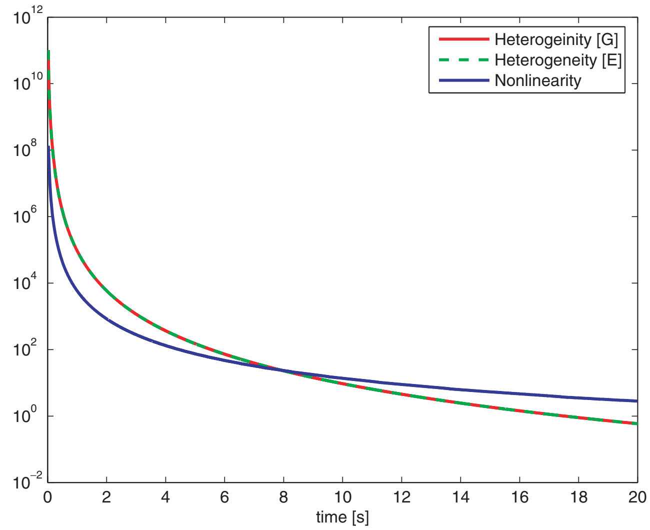

Figure 2. Envelope for scattered waves due to heterogeneities (Gaussian and exponential) and non-linearity of the media, computed at the distance of 0.1 km from the source. There is no significant difference in the envelope considering Gaussian or exponential autocorrelation functions for the heterogeneities.

The integral over space is treated as an integral over the isochrons (the surface where scattered waves arrive at the same time). More complicated source time functions can be considered but since at present we are interested in the main features of the envelope, this simple source is appropriate.

In Fig. 2 it is shown the envelope for both distributions of heterogeneities (Gaussian and exponential autocorrelation functions) and non-linearity of the media, computed for a distance of 0.1 km from the source, where interestingly we observe that: (1) the envelope due to the non-linear model tends to dominate at long time lapses over the heterogeneous models, (2) there is no major difference between Gaussian and exponential models of the heterogeneities at this range and frequency. The envelope, $e(t)$, is computed as $e(t) = \sqrt{u(t)^2 + H[u(t)]^2}$, where $u(t)$ and $H[u(t)]$ are the displacement and its Hilbert transform, respectively. We do observe that there is no significant difference for both cases of heterogeneties for distances at least up to 3 km.

Another aspect that influences the slope of the envelope is the source time function, which controls the radiation spectrum, and therefore the strength for different frequencies through the correlation function. A detailed comparison of different heterogeneous models, non-linear model and observations at different offsets will be presented elsewhere.

2.2.4 Examples

All seismic records show a coda, especially in the high-frequency range. An example is shown in Fig. 3 for the seismograms filtered between 1 and 10 Hz. and the envelopes in Fig. 4. These records are obtained from the TIPTEQ experiment (TIPTEQ 2006). The source of the seismograms is an explosion located at 20 m depth, and the stations are located at 0.1, 1.2 and 2.5 km away. The envelope decay of these observations (Fig. 4) depend on the distance to the source and the predominant frequency. Note that at the same distance, the amplitude decay changes slightly when changing the centre frequency of the bandpass filter in the range between 1 and 40 Hz. It is shown that at longer distances from the source, the decay is less pronounced, meaning that the energy is spread over a larger region compared to records at closer distances.

Synthetic envelopes are shown in Figs 5(a)–(c), superimposed on the observations. The main point is that the non-linear model fits quite good and even better than the heterogeneous model. For large distances, however, the non-linear model is not so good, but it reproduces the main features.

This is a very simple homogeneous infinite model, where we only include the effect of non-linearity of the elastic media, and we expect that considering a half space or layered structure could improve the model even more. It is reasonable to expect that the non-linear coefficients might vary through the media as well.

3 CONCLUSIONS

Most seismic records in the frequency range over 1 Hz show a significant amount of scattering which is explained as due to heterogeneities in the crust. This has been used to infer the characteristics of the assumed randomly distributed heterogeneities, as for the characteristic length of heterogeneities, and fluctuation of seismic velocities (Sato & Fehler 1997).

Here we show that the presence of heterogeneities is not the only explanation for such observations, and that the effect of weak non-linearity of the elastic media produces comparable observations.

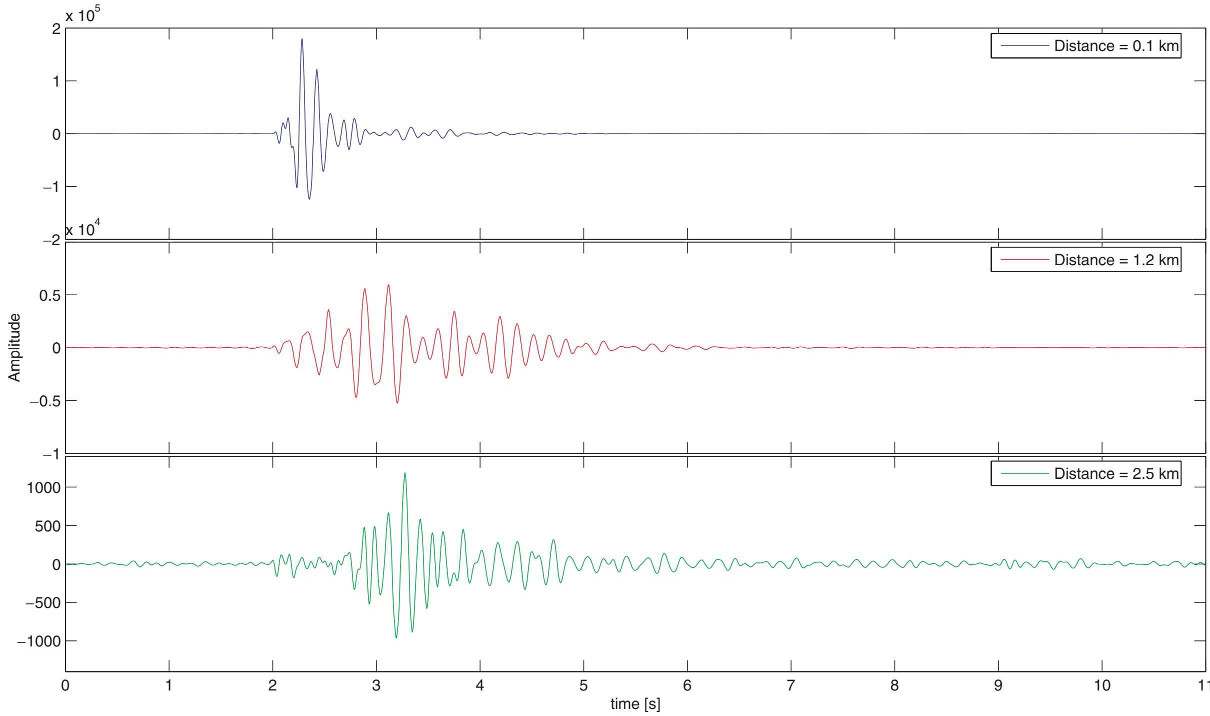

Figure 3. Observed seismograms filtered between 1 and 10 Hz recorded at distances of 0.1 (top), 1.2 (middle) and 2.5 (bottom) km. The records are shifted in time, beginning at the observed arrival time. At the closest station, the signal is very short (1 s), and it gets longer for stations farther away from the source.

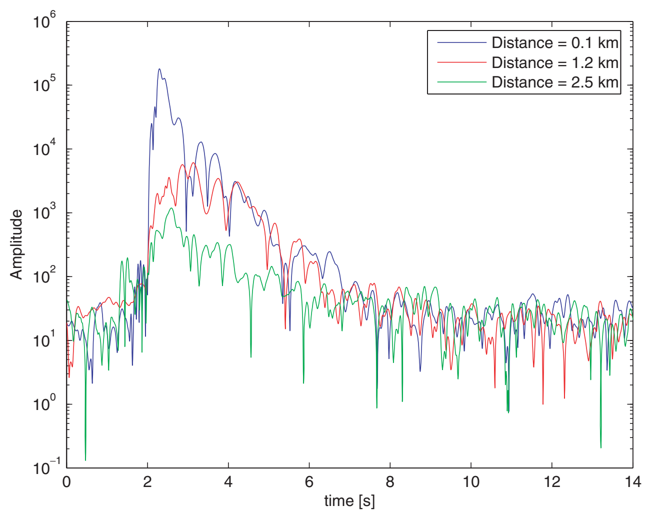

Figure 4. Envelopes of seismograms shown in Fig. 3. The envelopes are shifted in time, beginning at the observed arrival time. One can see that the amplitude decays smoothly during 5 s in all three cases.

The similarity of scattering due to heterogeneities and non-linearity is in the radiation pattern. The main difference between these two processes is the strength of the scattered field, which has a different dependence on the amplitude of the incoming wave and frequency.

For a non-linear process, the strength of the scattered field is proportional to the amplitude of the incoming wave up to the third power (when the strain energy depends up to fourth order in the strain tensor), while for a heterogeneous media, it is proportional to the first power. This implies that for the case of an earthquake, the strength of the scattered field will decay rapidly, simply because the amplitude of the incident wave decays as it propagates away from the source. This effect is not so strong in the heterogeneous medium.

We have used a special form of non-linearity as an example. There is no agreement upon which is more appropriate from the point of view of first principles and direct observations. Several forms have been proposed in the literature (Truesdell 1966), and it is not the intention in this paper to discuss this aspect. Further investigation to compare model predictions with data is underway and will be presented elsewhere. Here we want to show that simple non-linear elastic behaviour of rocks have an observable implication, and these observations can be used to infer about the characteristic of non-linearity, to understand further the behaviour of rock mechanics.

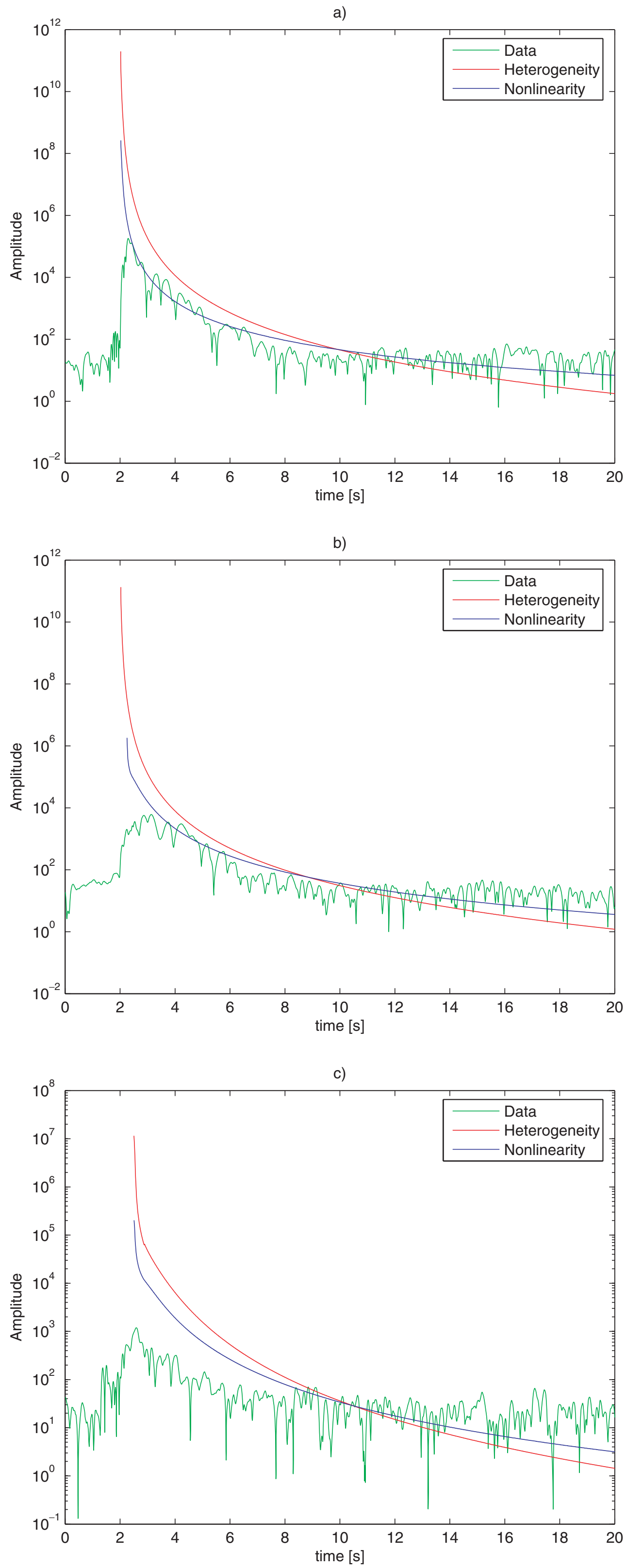

Figure 5. Model results compared with envelopes shown in Fig. 3 for the distance of (a) 0.1 km, (b) 1.2 km and (c) 2.5 km. The envelopes are shifted in time in order to have the same arrival time. One can see that the main features are well explained. There are discrepancies at the beginning of each trace, but these could be due to several factors including: (i) the degree of approximations used in the model, as the consideration of a source time function given by a delta function, or an infinite space and (ii) the contribution from heterogeneities.

ACKNOWLEDGMENTS

Thanks to J. Klotz for encouraging this research, S. Shapiro, P. Wigger and C. Krawczyk for fruitfull discussions. Funding from the University of Concepción (Grant 99.025.12), the German Ministry of Education and Research (BMBF) (Grant 03G0594), the German Research Foundation (DFG), the Freie Universitaet Berlin and the GFZ Potsdam, is appreciated. The seismic stations were provided by the Geophysical Instrument Pool Potsdam and the Freie Universitaet Berlin. I. Calisto was supported by MECESUP UCO0209. We acknowledge the suggestions by a reviewer and the editor Michael Korn.

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