Degeneracy in One-Dimensional Quantum Mechanics: A Case Study

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ABSTRACT: In this work we study the isotonic oscillator, $V(x) = Ax^2 + Bx^{-2}$, on the whole line $-\infty < x < +\infty$ as an example of a one-dimensional quantum system with energy level degeneracy. A symmetric double-well potential with a finite barrier is introduced to study the behavior of energy pattern between both limit: the harmonic oscillator (i.e., a system without degeneracy) and the isotonic oscillator (i.e., a system with degeneracy). Int J Quantum Chem 110: 1317–1321, 2010

Key words: isotonic oscillator; harmonic oscillator; double-well potential; degeneracy

1. Introduction

The problem of degeneracy of bound states in one-dimensional quantum mechanics has been studied over several decades [1–9] since Loudon showed the existence of degenerate states for one-dimensional hydrogen atom [10]. Basically, degeneracy is not allowed for the usual one-dimensional quantum systems by the well-known nondegeneracy Theorem [11]: if ψ_1 and ψ_2 are two solutions of the one-dimensional stationary Schrödinger equation

$$\psi''(x) + \frac{2\mu}{\hbar^2}[E - V(x)]\psi(x) = 0, \tag{1}$$

with $\psi_1(x)$ and $\psi_2(x)$ vanishing in the limit $x \to \pm\infty$, then the Wronskian of ψ_1 and ψ_2 functions is identically zero

$$W[\psi_1, \psi_2](x) = \psi_1(x)\psi_2'(x) - \psi_1'(x)\psi_2(x) = 0. \tag{2}$$

And integration of (2) then yields

$$\psi_2(x) = c\psi_1(x), \tag{3}$$

where c is an arbitrary constant. So, the wavefunctions ψ_1 and ψ_2 are linearly dependent and describes the same quantum state.

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Obviously, the above nondegeneracy Theorem is therefore not necessarily valid if $\psi_1(x)\psi_2(x) = 0$ anywhere. For instance, if we consider the following trial wave-functions $\phi_1(x) = x^2 e^{-x^2}$ and $\phi_2(x) = x|x|e^{-x^2}$, then it is easy to show that the Wronskian $W[\phi_1, \phi_2]$ is identically zero, but ϕ_1 and ϕ_2 are linearly independent on the whole real line $\mathbb{R}$. In other words, ϕ_1 and ϕ_2 are two degenerated wave-functions. Note, however, that ϕ_1 and ϕ_2 are linearly dependent if they are restricted either to the interval $\mathbb{R}^- \cup \{0\}$ or $\mathbb{R}^+ \cup \{0\}$, i.e.

$$\phi_2(x) = \begin{cases} c_+\phi_1(x), & x \in \mathbb{R}^+, \\ c_-\phi_1(x), & x \in \mathbb{R}^-, \end{cases} \tag{4}$$

with $c_+ = 1$ and $c_- = -1$. Result (4) is essentially the same as (3) for each connected components $\mathbb{R}^+$ and $\mathbb{R}^-$, respectively. Taking second derivative of ϕ's functions and replacing it in (1), we find the following expression for the potential function:

$$V(x) = Ax^2 + Bx^{-2}. \tag{5}$$

The above potential function is singular at origin $x = 0$ and it is an example of the main role played by singular potentials to define degeneracy in one-dimensional quantum mechanics [7]. In general, degeneracy could be allowed if the potential is singular at a node of the wave-functions.

Potential (5) was studied by Goldman and Krivchenkov [12]. They showed that the energy spectrum of this potential is an infinite set of equidistant energy levels similar to the harmonic oscillator. For this reason, this potential is termed *the isotonic oscillator* [13–18]. In this work, we consider the isotonic oscillator on the whole domain $-\infty < x < +\infty$ as a case study of a one-dimensional quantum system with energy level degeneracy. After a brief review of the isotonic oscillator in Section 2, a double-well model with a finite barrier is introduced in Section 3 to study the behavior of energy pattern between both limit cases: the harmonic oscillator (a system without degeneracy) and the isotonic oscillator (a system with degeneracy).

2. Isotonic Oscillator

Let us consider a particle of mass μ on the whole line $-\infty < x < +\infty$ under isotonic potential function

$$V(x) = \frac{1}{2}\mu\omega^2 x^2 + \frac{g}{x^2}, \tag{6}$$

where $\omega > 0$ and $g > 0$. After introducing the following dimensionless variables

$$y = \sqrt{\frac{\mu\omega}{\hbar}}x, \quad \varepsilon = \frac{E}{\hbar\omega}, \tag{7}$$

the Schrödinger equation for the isotonic oscillator reads

$$\psi''(y) + [2\varepsilon - v(y)]\psi(y) = 0, \tag{8}$$

where

$$v(y) = y^2 + \frac{1}{4}(2\lambda + 1)(2\lambda - 1)\frac{1}{y^2}, \tag{9}$$

and

$$\lambda = \frac{1}{2}\sqrt{1 + \frac{8\mu g}{\hbar^2}}. \tag{10}$$

In particular, harmonic oscillator is recovered in the limit $g \to 0$ (i.e., $\lambda \to 1/2$). The constrain $0 < g$ restricts λ values to $1/2 < \lambda$. The energy levels for the isotonic oscillator are given by

$$\varepsilon_n = 2n + 1 + \lambda, \quad n = 0, 1, 2, \ldots \tag{11}$$

and the unnormalized wave-functions are

$$\psi_n^{(\text{even})}(y) = \begin{cases} y y^{\lambda-\frac{1}{2}} e^{-y^2/2} F(-n, 1+\lambda, y^2) & \text{for } y \geq 0 \\ -y|y|^{\lambda-\frac{1}{2}} e^{-y^2/2} F(-n, 1+\lambda, y^2) & \text{for } y < 0 \end{cases} \quad \text{with } n = 0, 1, 2, \ldots, \tag{12}$$

and

$$\psi_n^{(\text{odd})}(y) = \begin{cases} y y^{\lambda-\frac{1}{2}} e^{-y^2/2} F(-n, 1+\lambda, y^2) & \text{for } y \geq 0 \\ y|y|^{\lambda-\frac{1}{2}} e^{-y^2/2} F(-n, 1+\lambda, y^2) & \text{for } y < 0 \end{cases} \quad \text{with } n = 0, 1, 2, \ldots, \tag{13}$$

where $F(\alpha, \gamma, z)$ is the confluent hypergeometric function. The normalization constant for the isotonic oscillator wave-functions is

$$N_n = \sqrt{\frac{(\lambda+1)(\lambda+2)\cdots(\lambda+n)}{n!\Gamma(\lambda+1)}\sqrt{\frac{\mu\omega}{\hbar}}}. \tag{14}$$

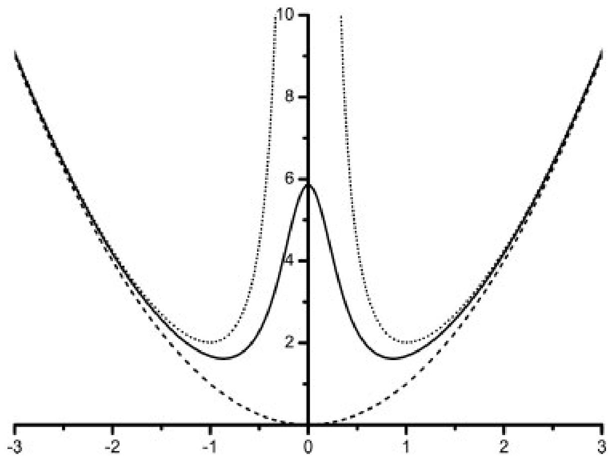

FIGURE 1. The potential $y^2 + b/(\kappa + y^2)$ is plotted for $b = 3/4$ ($\lambda = 1$) and $\kappa = 0$ (dots), $\kappa = 0.125$ (line) and $\kappa \to \infty$ (dash).

3. Double-Well Model

To study the behavior of energy pattern between both limit cases: the harmonic oscillator (a system without degeneracy) and the isotonic oscillator (a system with degeneracy), let us consider the following double-well potential function

$$w(y) = y^2 + \frac{b}{\kappa + y^2}, \tag{15}$$

where

$$b = \frac{1}{4}(2\lambda + 1)(2\lambda - 1). \tag{16}$$

Potential function (15) reduces to that of the isotonic oscillator in the limit $\kappa \to 0$ and to that the harmonic oscillator in the limit $\kappa \to \infty$. Figure 1 shows the behavior of potential (15) for several values of κ parameter.

Stationary Schrödinger equation for a particle of mass μ oscillating on the whole real line $-\infty < x < +\infty$ under double-well potential (15) reads

$$(\kappa + y^2)\left[D_{yy} - y^2\right]\psi(y) + [2\varepsilon(\kappa + y^2) - b]\psi(y) = 0. \tag{17}$$

We can write the function ψ in (17) in terms of the complete orthonormal system of eigenfunctions of the corresponding harmonic oscillator, i.e.

$$\psi(y) = \sum_{n=0}^{\infty} c_n \psi_n(y), \tag{18}$$

where $\psi_n(y)$ are the so-called Weber–Hermite functions [19]. So, we can use the properties of Weber–Hermite functions,

$$\left[D_{yy} - y^2\right]\psi_n(y) = -(2n+1)\psi_n(y), \tag{19}$$

and

$$y\psi_n(y) = n\psi_{n-1}(y) + \frac{1}{2}\psi_{n+1}(y), \tag{20}$$

to obtain the following recursion system for the c's coefficients:

$$\alpha_m c_{m-2} + \beta_m c_m + \gamma_m c_{m+2} = 0, \tag{21}$$

where

$$\alpha_m = (2m - 3 - 2\varepsilon)/4, \tag{22}$$
$$\beta_m = (2m + 1 - 2\varepsilon)(2\kappa + 2m + 1)/2 + b, \tag{23}$$
$$\gamma_m = (2m + 5 - 2\varepsilon)(m + 2)(m + 1), \tag{24}$$

for $m = 0, 1, 2, \ldots$ with $\alpha_0 = \alpha_1 = 0$ by definition. Because of the symmetry of potential (15) under reflection $y \to -y$, even and odd solutions can be obtained by choosing $c_0 \neq 0$ and $c_1 = 0$ or $c_0 = 0$ and $c_1 \neq 0$, respectively. In order that the system (21) may posses nontrivial solutions, the associated tridiagonal determinants for even and odd solutions, respectively, must vanish, i.e.,

$$\begin{vmatrix} \beta_0 & \gamma_0 & 0 & 0 & \cdots \\ \alpha_2 & \beta_2 & \gamma_2 & 0 & \cdots \\ 0 & \alpha_4 & \beta_4 & \gamma_4 & \cdots \\ 0 & 0 & \alpha_6 & \beta_6 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{vmatrix} = 0$$

$$\text{and} \begin{vmatrix} \beta_1 & \gamma_1 & 0 & 0 & \cdots \\ \alpha_3 & \beta_3 & \gamma_3 & 0 & \cdots \\ 0 & \alpha_5 & \beta_5 & \gamma_5 & \cdots \\ 0 & 0 & \alpha_7 & \beta_7 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{vmatrix} = 0. \tag{25}$$

For a given m, it is straightforward to see, from (22) to (24), that in the limit $\kappa \to \infty$ we have $\alpha_m/\kappa \sim 0$, $\beta_m/\kappa \sim (2m + 1 - 2\varepsilon)$, and $\gamma_m/\kappa \sim 0$. So, condition

TABLE I

Eigenvalues ε for several values of log(κ) with $\lambda = 1$, after solving numerically Eq. (25).

$\log \kappa$	ε_0	ε_1	ε_2	ε_3	ε_4	ε_5	ε_6	ε_7	ε_8	ε_9
−4.000	1.9958	1.9961	3.9914	3.9921	5.9868	5.9881	7.9821	7.9840	9.9771	9.9799
−3.500	1.9948	1.9957	3.9896	3.9914	5.9843	5.9872	7.9787	7.9829	9.9730	9.9786
−3.000	1.9920	1.9945	3.9842	3.9894	5.9762	5.9845	7.9681	7.9796	9.9597	9.9748
−2.500	1.9830	1.9908	3.9668	3.9833	5.9503	5.9763	7.9335	7.9696	9.9162	9.9632
−2.000	1.9544	1.9803	3.9099	3.9659	5.8636	5.9533	7.8152	7.9418	9.7651	9.9313
−1.500	1.8646	1.9532	3.7212	3.9233	5.5723	5.8990	7.4308	7.8785	9.3067	9.8607
−1.000	1.6021	1.8951	3.2465	3.8418	5.0288	5.8042	6.9094	7.7759	8.8392	9.7537
−0.500	1.1427	1.7971	2.8267	3.7312	4.7326	5.6935	6.6895	7.6689	8.6640	9.6514
0.000	0.7811	1.6792	2.6542	3.6314	4.6188	5.6080	6.6003	7.5937	8.5885	9.5839
0.500	0.6051	1.5849	2.5737	3.5661	4.5604	5.5560	6.5524	7.5495	8.5470	9.5448
1.000	0.5359	1.5329	2.5306	3.5288	4.5272	5.5259	6.5247	7.5237	8.5228	9.5220
1.500	0.5117	1.5113	2.5110	3.5107	4.5105	5.5102	6.5100	7.5098	8.5096	9.5094
2.000	0.5037	1.5037	2.5037	3.5037	4.5036	5.5036	6.5035	7.5035	8.5035	9.5034
2.500	0.5012	1.5012	2.5012	3.5012	4.5012	5.5012	6.5012	7.5012	8.5012	9.5012
3.000	0.5004	1.5004	2.5004	3.5004	4.5004	5.5004	6.5004	7.5004	8.5004	9.5004
3.500	0.5001	1.5001	2.5001	3.5001	4.5001	5.5001	6.5001	7.5001	8.5001	9.5001
4.000	0.5000	1.5000	2.5000	3.5000	4.5000	5.5000	6.5000	7.5000	8.5000	9.5000

(25) implies $\varepsilon \sim m + 1/2$, i.e., the harmonic oscillator limit. In general, for given values of λ and κ parameters, ε can be evaluated after solving numerically the above tridiagonal determinant conditions (25). Table I shows the numerical results of ε for several values of $\log(\kappa)$ with $\lambda = 1$. The plot of these results (see Fig. 2) shows the smooth transition of energy pattern from nondegeneracy to degeneracy.

4. Conclusions

Isotonic oscillator, defined over the whole domain $-\infty < x < +\infty$, is an illustrative example of a one-dimensional quantum system with a singularity (located at the origin) that exhibits energy degeneracy. At the singular point, both even and odd isotonic oscillator wave-functions have a node at origin and its first derivative there exists, so the nondegeneracy Theorem is overcame because every pair of solutions ψ and φ for a given energy satisfy the condition

$$\psi(x)\varphi'(x) - \psi'(x)\varphi(x) = 0,$$

without leading to the conclusion of linear dependence of ψ and φ. The term $y|y|^{\lambda-\frac{1}{2}}$ in the isotonic degenerated wave-functions is responsible of continuity at the origin (the singular point) of each wave-function and its first derivative. Certainly, the above arguments are not present in case of harmonic oscillator and degeneracy is not allowed for that system. A smooth transition from degeneracy (isotonic oscillator) to nondegeneracy (harmonic oscillator) was studied by the introduction of a symmetric double-well potential with finite barrier.

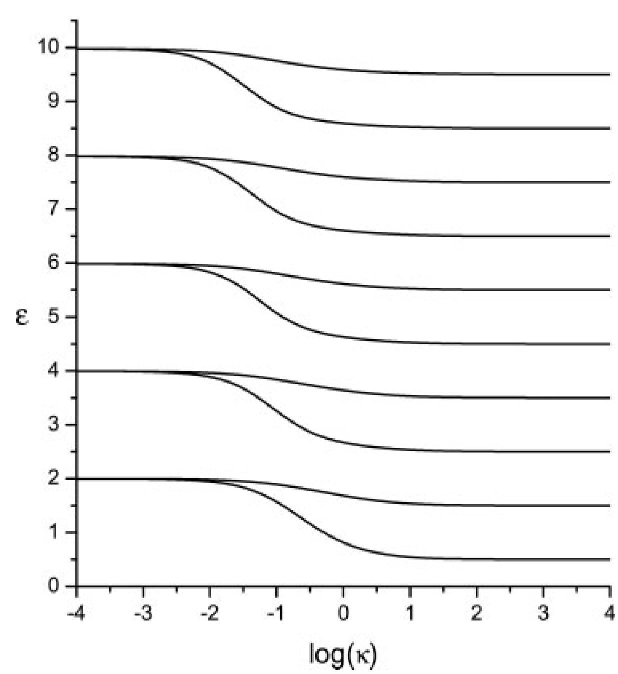

FIGURE 2. Eigenvalues ε as a function of $\log(\kappa)$ are plotted for the double-well model with $\lambda = 1$.

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References

1. Andrews, M. Am J Phys 1966, 34, 1194.
2. Haines, L. K.; Roberts, D. H. Am J Phys 1969, 37, 1145.
3. Andrews, M. Am J Phys 1976, 44, 1064.
4. Kwong, W.; Rosner, J. L.; Schonfeld, J. F.; Quigg, C.; Thacker, H. B. Am J Phys 1980, 48, 926.
5. Hammer, C. L.; Weber, T. A. Am J Phys 1988, 56, 281.
6. Cohen, J. M.; Kuharetz, B. J Math Phys 1993, 34, 12.
7. Bhattacharyya, K.; Pathak, R. K. Int J Quantum Chem 1996, 59, 219.
8. Koley, R.; Kar, S. Phys Lett A 2007, 363, 369.
9. Kar, S.; Parwani, R. R. Europhys Lett 2007, 80, 30004.
10. Loudon, R. Am J Phys 1959, 27, 649.
11. Landau, L. D.; Lifshitz, E. M. Quantum Mechanics; Pergamon Press: Oxford, 1977.
12. Goldman, I. I.; Krivchenkov, V. D. Problems in Quantum Mechanics; Pergamon Press: London, 1961; Chapter 1 (Problem 11).
13. Weissman, Y.; Jortner, J. Phys Lett A 1979, 70, 177.
14. Dongpei, Z. J Phys A 1987, 20, 4331.
15. Cariñena, J. F.; Rañada, M. F.; Santander, M. SIGMA 2007, 3, 030.
16. Asorey, M.; Cariñena, J. F.; Marmo, G.; Perelomov, A. M. Ann Phys 2007, 322, 1444.
17. Cariñena, J. F.; Rañada, M. F.; Perelomov, A. M. J Phys Conf Ser 2007, 87, 012007.
18. Cariñena, J. F.; Perelomov, A. M.; Rañada, M. F.; Santander, M. J Phys A 2008, 41, 085301.
19. Bell, W. W. Special Functions for Scientist and Engineers; Dover: New York, 2004; Chapter 5.