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Evidence that non-linear elasticity contributes to the seismic coda

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SUMMARY

Different factors might affect the propagation of seismic waves producing scattering, including heterogeneities and non-linear elasticity. A key difference between these two factors is the dependence of the strength of the scattered waves on the strength of the incident wave, being linear for the former and non-linear for the latter. A detailed study of the TIPTEQ data, where about a hundred explosions were recorded on 180 three-component stations in the distance range of approximately 0-100 km, shows that this dependence is non-linear. Data were analyzed in the following way: (i) the envelope of bandpass filtered data between 10 and 40 Hz was obtained for a large number of stations from different distance ranges and charge sizes of shots, (ii) for these distances we modeled the envelope considering the non-linear elasticity. The shapes of the theoretical and observed envelopes were in general very similar. A scale factor for each case was obtained considering the best fit of its complete envelope, and (iii) since this scale factor depends mainly on the size of the explosion, we computed the ratio (R) of the scale factor (sf) for different sizes of explosions at the same distance. Finally, varying the distance between 0 and 50 km, (iv) we computed the power (p) of the dependence of the ratio (R) on the ratio of the charge sizes ($R = (\frac{sf_1}{sf_2}) = (\frac{charge_1}{charge_2})^p$). For the complete data set we obtain a value of $p = 2.5 \pm 0.9$, which is clearly greater than 1. This shows that non-linear elasticity is an important factor in the contribution to seismic wave scattering in the frequency range of 10-40 Hz.

Key words: Theoretical Seismology; Wave Propagation; Wave Scattering and Diffraction.

1 INTRODUCTION

The principal contribution of scattering in coda waves was originally thought to be heterogeneities, which are present in the crust. The coda wave is the most worthwhile to study when considering scattering as it has the highest sensitivity to the changes in the media (Nikolaev, 1987). However, since laboratory experiments show that a non-linear behavior of rocks exists close to the rupture condition (Sholz, 1990) then the presence of heterogeneities would not be the only explanation for the scattering in the envelope of seismic waves, and the effect of weak non-linearity of the elastic medium produces comparable observations even when the medium is homogeneous (Bataille and Calisto, 2008).

The interpretation of seismic results based on Hooke's model fails to account for certain observed effects (Lyakhovsky and Myasnikov, 1988) such as, for example, a special experimental investigation which allowed the propagation of periodic seismic signals using seismic vibrators. During this investigation strong non-linear effects that are caused by physical non-linearity of the medium were observed (Nikolaev, 1988).

One can point out that, in general, how the amplitude of the scattered waves scales with the seismic moment depends directly on the constitutive law of the media. For the non-linear process, there is still no agreement upon the most appropriate form for the strain energy from the point of view of physical principles and direct observations.

The derivation of the equations of motion for the displacement field depends on the expression of the strain energy, which is not uniquely defined, since different forms have been proposed. For instance, to third order in the strain tensor, was presented by McCall (1994), and one for the strain energy, depending up to fourth order on the strain tensor, by Bataille and Calisto (2008). In the latter work the theoretical strength of the scattered field for this special form of strain energy is shown to be proportional to the amplitude of the incoming wave up to the third power, while for a heterogeneous medium it is proportional to the first power. Understanding the appropriate constitutive law to be used for the crust, or the Earth in general, will depend strictly on observations. This work is a first step in this direction.

By following this previous work we analyse the data for explosions with different initial amplitudes at different source-receiver distances, and show the evidence that the non-linearity of the elastic media contributes to the scattering of the seismic coda. We start with a brief summary of the theory, secondly the modeling of the data, considering two different temporal functions, and finally we finish by showing evidence for the non-linearity.

2 THEORY

We consider a general non-linear elastic constitutive relationship derived from $\tau = \frac{\partial W}{\partial \varepsilon}$, where the elastic energy function $W(I_1, I_2, I_3)$ depends only on the strain (ε) through its invariants $I_1 = Trace(\varepsilon)$, $I_2 = Trace(\varepsilon \cdot \varepsilon)$ and $I_3 = Det(\varepsilon)$. Up to fourth order in terms of strain, the elastic energy is

$$W(I_1, I_2, I_3) = (aI_1^2 + bI_2 + cI_1^4 + dI_2^2 + eI_1^2 I_2)/2 \qquad (1)$$

where a and b relate to the Lamé constants, and c, d and e are non-linear parameters of the media, describing the departure from Hooke's law. The second order term of strain relates to the linear elastic theory, while the third order is an odd function which we disregard because the energy should be positive defined.

By using the constitutive equation in the equation of motion $\rho \ddot{u}_i = \partial_j \tau_{ij} + f_i$, we obtain the resultant elastodynamic equation,

$$\begin{aligned} \rho \ddot{u}_i - a\varepsilon_{pp,i} - b\varepsilon_{ij,j} &= f_i + 6c\varepsilon_{pp}\varepsilon_{qq}\varepsilon_{kk,i} + 2d\left\{\varepsilon_{pq}\varepsilon_{qp}\varepsilon_{ij,j} + 2\varepsilon_{pq}\varepsilon_{qp,j}\varepsilon_{ij}\right\} \qquad (2) \\ &\quad + e\left\{\varepsilon_{pq}\varepsilon_{qp}\varepsilon_{kk,i} + \varepsilon_{kk}\varepsilon_{ll}\varepsilon_{ij,j} + 2\varepsilon_{kk}\varepsilon_{pq,i}\varepsilon_{qp} + 2\varepsilon_{kk}\varepsilon_{ll,j}\varepsilon_{ij}\right\} \end{aligned}$$

where analytical solutions for this equation are not possible.

Solutions can be found using the perturbation approach by assuming that the complete solution is close to the linear solution. The difference, depending weakly on the non-linearity, is thus

$$
\begin{aligned}
u &= u^l + u^{nl} \ ; \qquad u^{nl} \ll u^l \\
\varepsilon &= \varepsilon^l + \varepsilon^{nl} \ ; \qquad \varepsilon^{nl} \ll \varepsilon^l
\end{aligned}
$$

Replacing these into the elastodynamic equation (2) we have

$$
\begin{aligned}
\rho \ddot{u}_i^l - a\varepsilon_{pp,i}^l - b\varepsilon_{ij,j}^l &= f_i \\
\rho \ddot{u}_i^{nl} - a\varepsilon_{pp,i}^{nl} - b\varepsilon_{ij,j}^{nl} &= F_i^{nl}(\varepsilon^l) + O\left((\varepsilon^l)^2\varepsilon^{nl}, \varepsilon^l(\varepsilon^{nl})^2, (\varepsilon^{nl})^3\right) \qquad (3)
\end{aligned}
$$

The first equation of (3) is recognized as the linear elastodynamic equation where $a = \lambda$ and $b = \mu$ are the Lamé constants. For the second equation of (3), we can neglect O compared to F^{nl} up to first order of the non-linearity, since $\varepsilon^{nl} \ll \varepsilon^l$. Thus, to first order, the equation for u^{nl} becomes the linear elastodynamic equation, where all non-linear terms are considered as equivalent forces within a linearly homogeneous medium. The equivalent force is given by,

$$
\begin{aligned}
F_i^{nl} &= 6c\varepsilon_{pp}^l\varepsilon_{qq}^l\varepsilon_{kk,i}^l + 2d\left(\varepsilon_{kl}^l\varepsilon_{lk}^l\varepsilon_{ij,j}^l + 2\varepsilon_{kl}^l\varepsilon_{lk,j}^l\varepsilon_{ij}^l\right) \\
&\quad + e\left(\varepsilon_{pq}^l\varepsilon_{qp}^l\varepsilon_{kk,i}^l + \varepsilon_{pp}^l\varepsilon_{kk}^l\varepsilon_{ij,j}^l + 2\varepsilon_{pp}^l\varepsilon_{lk}^l\varepsilon_{kl,i}^l + 2\varepsilon_{pp}^l\varepsilon_{ij}^l\varepsilon_{kk,j}^l\right) \qquad (4)
\end{aligned}
$$

This equivalent force represents the source of scattering due to non-linear elasticity.

To compare theory with observations, we have to compute the contribution to scattered waves due to this effect. Since the observations used in this paper relate to the active seismic experiment TIPTEQ (Gross et al, 2008), where shots are chemical explosions, let us consider in our model an explosion point source, $f_i = -\partial_i M$, where M is the scalar moment tensor. In this case, equation (4) is reduced to

$$
F_i^{nl} = \left(\frac{6(c+d+e)}{\alpha}\right)\left(\frac{1}{4\pi\alpha^4 r}\right)^3 \ddot{M^\alpha}^2 \dddot{M}^\alpha \gamma_i \qquad (5)
$$

where α is the P-wave velocity, $\gamma_i = \frac{x_i}{|x|}$, $r = |x|$, $M^\alpha = M(t - \frac{r}{\alpha})$. The non-linear coefficient $6(c+d+e)$ we will denote Λ.

We observe in equation (5) that the force is proportional to the moment (M) to the power of 3, which is different to the case of scattering due to heterogeneities (Aki & Richards, 2002), in which case the power is 1. This difference should be observable, and will be used in this paper as the main discriminant between scattering due to heterogeneities and non-linearity.

When we compare this result with that for scattering due to heterogeneities of the medium we see that the heterogeneities have a linear dependency on the seismic moment ($\frac{\dddot{M}(t)}{r}$) and the non-linearities depend on the third power as

$$u^{obs} \propto \frac{\ddot{M}(t)^2 \dddot{M}(t)}{r^3} \tag{6}$$

Once the expression for the source of scattering is determined, the scattering solution for the displacement is given by (Aki & Richards, 2002)

$$u_i^{nl}(x,t) = \int \int G_{ij}(x,x',t,t) F_j^{nl}(x',t') dx' dt' \tag{7}$$

where $G_{ij}(x,x',t,t)$ is the Green's function.

3 MODELING SCATTERED WAVES

3.1 Envelope

To compute the envelope of scattered waves due to non-linearity of the media we use eq. (7) where the integral over the space is treated as an integral over isochrons (the surface where the scattered waves arrive at the same time). The scattered waves are given by

$$\begin{aligned} u_i^{nl}(r,t) &= C_p \int \frac{\gamma_i \gamma_j \gamma_j'}{|h||r'|^3} \left[\ddot{M}\left(t - \frac{|r'|}{\alpha} - \frac{|h|}{\alpha}\right)\right]^2 \dddot{M}\left(t - \frac{|r'|}{\alpha} - \frac{|h|}{\alpha}\right) dV' \\ &+ C_s \int \frac{(\gamma_i' - \gamma_i \gamma_j \gamma_j')}{|h||r'|^3} \left[\ddot{M}\left(t - \frac{|r'|}{\alpha} - \frac{|h|}{\beta}\right)\right]^2 \dddot{M}\left(t - \frac{|r'|}{\alpha} - \frac{|h|}{\beta}\right) dV' \end{aligned}$$

with $h_i = r_i - r_i'$ and

$$\begin{aligned} C_p &= \left(\frac{\Lambda}{4\pi\rho\alpha^3}\right)\left(\frac{1}{4\pi\rho\alpha^4}\right)^3 \\ C_s &= \left(\frac{\Lambda}{4\pi\rho\beta^2}\right)\left(\frac{1}{4\pi\rho\alpha^4}\right)^3 \end{aligned}$$

The simplest source time function is a Dirac delta, which simplifies computations. However, to include the history of the source's rupture and finiteness, it is convenient to use a finite source time function, and for simplicity we use a Gaussian function. Both cases are shown in Figure (1). In a) the comparison of both during the first 20 seconds is shown where it is not possible to see a significant difference between them. In b) the first 3 seconds are shown where we can notice some

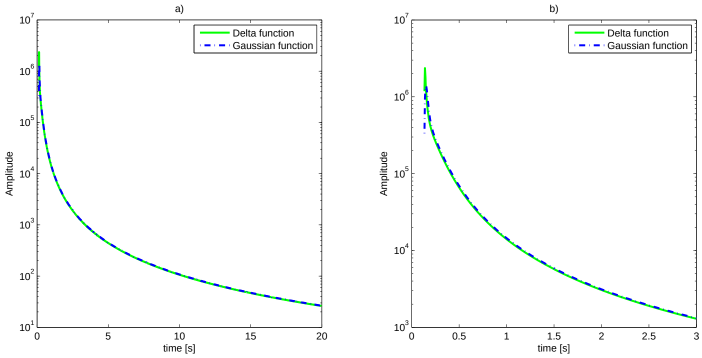

Figure 1. a) Comparison between delta temporal function (green line) and convolution with Gaussian function (blue dotted line). b) Zoom in of a) on the first 3 seconds where there is the most difference between the functions.

difference. Since this difference is small, we conclude that it is appropiated to use the Dirac delta function as a source time function, and this will be used in the following computations.

Figure (2) shows the fit of the model to seismic coda waves using a Dirac delta as a temporal function. The data were obtained from the TIPTEQ experiment (Gross et al, 2008), where the source of the seismograms is an explosion located at 20 m depth. Every seismogram was filtered with a bandpass filter between 10 and 40 Hz.

3.2 Scale

The scattered wave due to heterogeneities is linearly proportional to the moment and therefore to the initial amplitude. In contrast, by considering non-linear elasticity the scattered wave depends on the third power of the moment (see eq. (6)). To compare the model with data we have to scale the model. We can express the observed wave as

$$u^{obs}(x,t;q) = D\ u^{nl}(x,t;q) \tag{8}$$

where q is the charge of the explosion, u^{nl} is the model and D is a constant of proportionality. We can write the model as

$$u^{nl}(x,t;q) = q^{p}\, u(x,t) \tag{9}$$

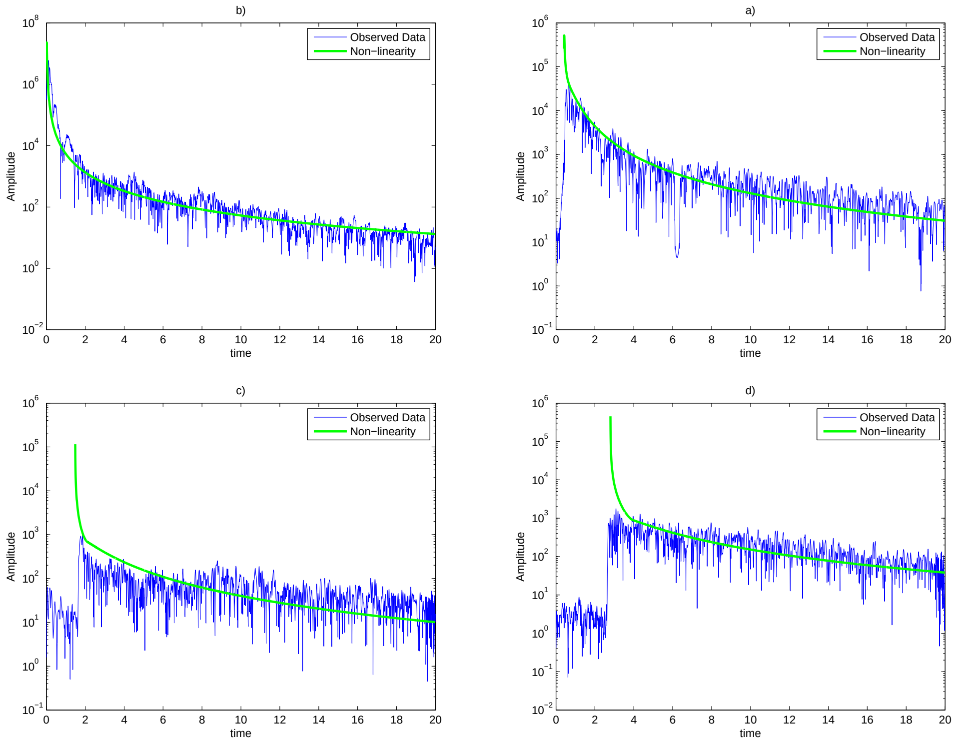

Figure 2. Envelope seismograms filtered between 10 and 40 Hz. a) Shot FFID 81 of 75 kilograms at 47 m source-receiver distance. b) Shot FFID 37 of 75 kilograms at 2065 m source-receiver distance. c) Shot FFID 31 of 100 kilograms at 7400 m source-receiver distance. d) Shot FFID 56 of 150 kilograms at 13980 m source-receiver distance

The power p represents the linear or non-linear behavior depending on the value. For $p = 1$ it would be linear but if it is different to 1 we will have a non-linear behavior. Our special relationship of strain energy has $p = 3$ (eq. (5)).

We analyzed the data by calculating the envelope of bandpass filtered data between 10 and 40 Hz of a large number of stations from different distance ranges and charge sizes of shots. Then we consider the fit of the real data to the model and obtain a scale factor (sf) for each case, i.e,

$$sf = D\ q^p \tag{10}$$

To prove this non-linear behavior we use the sf to calculate p by comparing two shots with different charge size at the same source-receiver distance. For example, for a shot with a charge of 75 kg we have by combining (8) and (9)

$$u^{obs} = D\ 75^p\, u_{75}(x,t) \tag{11}$$

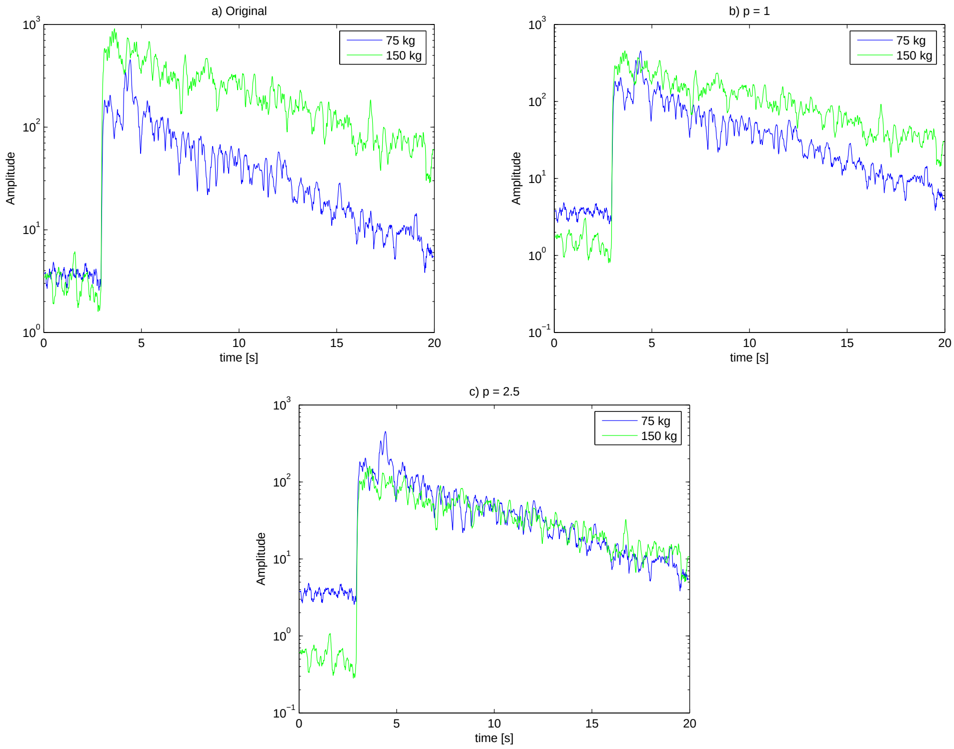

Figure 3. Envelope seismograms of two different shots (FFID 56 and 60) at 13.98 km source-receiver distance, where a) are the originals, b) are normalized linearly and c) are normalized with $p = 2.5$.

and for a 150 kg shot

$$u^{obs} = D\ 150^{p}\, u_{150}(x,t) \tag{12}$$

Comparing (11) and (12) we obtain the ratio R as

$$R = \left(\frac{75}{150}\right)^{p} = \frac{sf_{75}}{sf_{150}} \tag{13}$$

As a result we have a big difference when we consider $p = 1$ and a p different from unity. This is shown in Figure (3) where an example of non-linear dependence of the wave is observed with respect to the initial amplitude. Figure (3a) shows the seismogram of two different shots, with charge sizes of 75 and 150 kg respectively. Figure (3b) is the result when we multiply the data considering $p = 1$. Finally, Figure (3c) shows the multiplication with $p = 2.5$. These two shots were recorded at a distance of 13.98 km.

Using eq. (13) we can obtain from the data set, the variation of p as a function of source-receiver distance, which is shown in Figure (4), where we can see that the values of p are different to unity expected from linear behavior. The values of p used in the figure are shown in the ap-

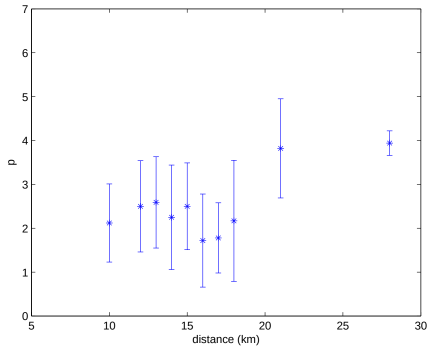

Figure 4. Power p as a function of the source-receiver distance for different shots.

pendix (Table A1). Every point in this figure is an average value for each distance. We made the calculations by analyzing over 150 seismograms obtaining around 70 values of p

4 CONCLUSION

The power p indicates whether the strength of scattered waves in the seismic coda is mainly due to heterogeneities ($p = 1$), or to non-linear elasticity of the medium ($p > 1$). This study shows that for different source-receiver distances, the scattering is due to non-linear elasticity, because $p = 2.5 \pm 0.9$ which differs significantly from the value of 1 expected from heterogeneities.

REFERENCES

Aki K. & Richards P.G., 2002. *Quantitative Seismology*, 2nd edn, University Science Books, Sausalito, CA, USA, 700 p.

Bataille K. & Calisto I., 2008. Seismic coda due to non-linear elasticity, *Geophys. J. Int.* **172**,2, 572-580.

Bataille K. & Lund F,. 1982. Non-linear waves in elastic media, *Physica D*. **6**, 95-104.

Bell J.F., 1984 *Mechanics of Solids*, Vol 1: The Experimental Foundations of Solid Mechanics, Springer-Verlag, Berlin, 813 p.,

Birch F., 1961. The velocity of compressional waves in rocks to 10 kilobars, *J. geophys. Res.*, **66**, 2199-2224.

Gross, K., Micksch, U. and TIPTEQ Research Group, Seismics Team, 2008. The reflection seismic survey of project TIPTEQ - the inventory of the Chilean subduction zone at 38.2^o S. *Geophys. J. Int.*. **172**, 565-571.

Johnson P.A. & McCall K.R., 1994. Observation and implications of non-linear elastic wave response in rock. *Geophys. Res. Lett.* **21**,165-168.

Lyakhovsky V.A. and Myasnikov V.P., 1988. Acoustics of rheologically non-linear solids, *Phys. Earth Planet Inter.*, **50**, 60-64.

McCall, K.R., 1994. Theoretical study of non-linear elastic propagation, *J. Geophys. Res.*, **99**, 2591-2600.

Murnagham, F.D., 1951. *Finite Deformation of an Elastic Solid*, 1st edn, John Wiley and Sons, Inc, New York.

Nikolaev A.V., 1988. Problems of nonlinear seismology, *Phys. Earth Planet Inter.*, **50**, 1-7.

Okada, Y., 1985. Surface deformation to shear and tensile faults in a half-space, *Bull. seis. Soc. Am.*, **75**, 1135-1154.

Rutherford, S.R. & Hawker, K.E., 1981, Consistent coupled mode theory of sound propagation for a class of non-separable problems, *J. acoust. Soc. Am.*, **71**, 554-564.

Sato, H. & Fehler, M., 1997. *Seismic Wave Propagation and Scattering in the Heterogeneous Earth*, Springer-Verlag, New York.

Scholz, C.H., 1990. *The Mechanics of Earthquakes and Faulting*, Cambridge University Press, Cambridge, UK, 439 p.

Truesdell, C., 1996. *The Mechanical Foundation of Elasticity and Fluid Dynamics*, Gordon and Breach Science Publishers, New York, 212 p.

Winkler, K., Nur, A. & Gladwin, M., 1979. Friction and seismic attenuation in rocks, *Nature*, **277**, 528-531.

Wu, R.S. & Aki, K., 1985. Scattering characteristics of elastic waves by an elastic heterogeneity, *Geophysics*, **50**, 582-595.

Zheng, H., Zhang, A. & Liu, E., 2006. Non-linear seismic wave propagation in anistropic media using the flux-corrected transport technique, *Geophys. J. Int*, **165**,943-956.

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APPENDIX A:

The table presents the source-receiver distance for different shots with different charge sizes, which charge sizes were compared and the value of p obtained.

Table A1. List of source-receiver distances and charge sizes used to calculate p

Distance (km)	Charges (kg)	p	Distance (km)	Charges (kg)	p
10.09	75,100	3.10	16.20	75,150	4.68
10.29	75,100	1.91	16.20	75,100	3.32
10.40	75,100	1.36	16.20	100,150	0.97
12.17	75,100	3.68	16.40	75,150	1.36
12.65	75,100	2.16	16.40	100,150	1.44
12.97	75,100	1.68	16.41	75,150	1.22
13.28	75,100	3.94	16.40	75,100	1.25
13.60	75,150	1.74	16.40	75,100	0.91
13.60	75,150	0.67	16.61	75,100	1.15
13.60	75,150	2.51	16.60	75,150	2.07
13.60	75,150	1.51	16.60	75,150	1.52
13.60	100,150	4.14	16.61	75,150	1.19
13.70	75,150	1.55	16.61	100,150	2.60
13.70	75,150	1.61	16.83	75,150	1.20
13.78	75,100	2.11	16.83	75,150	0.85
13.78	75,150	3.38	17.15	75,150	1.38
13.78	100,150	4.29	17.38	75,150	0.85
13.90	100,150	2.39	17.57	75,150	1.12
13.90	100,150	2.79	17.72	75,150	3.01
13.98	75,100	2.94	17.92	75,150	2.13
13.98	75,150	2.50	17.92	75,150	2.17
13.98	75,150	3.84	18.05	75,150	2.09
13.98	100,150	2.18	18.05	75,150	1.24
14.20	75,100	2.59	18.05	100,150	4.91
14.20	75,150	1.92	18.25	75,100	1.22
14.20	100,150	1.45	18.25	75,150	1.35
14.40	75,100	3.91	18.25	75,150	0.90